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Fourier Transform Reconstruction from Inexact Data

C. L. BYRNE AND R. M. FITZGERALD

Applied Ocean Acoustics Branch
Acoustics Division

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Methods for the linear reconstruction of Fourier transforms based on approximation-theoretic techniques are extended to the case of imprecise samples. Statistics of these estimators in the presence of random noise are calculated, and the special case of sinusoids in noise is discussed. Biased estimates of sinusoid amplitudes are obtained and related to the best linear unbiased estimates (BLUE).

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CONTENTS

FOURIER TRANSFORM RECONSTRUCTION FROM INEXACT DATA

INTRODUCTION

The central problem in spectral estimation, the reconstruction of a Fourier transform from sampled data, is, to paraphrase Parzen [1], essentially the problem of how best to approximate a function from N of its Fourier coefficients. Emphasis on the approximation-theoretic aspect of the problem focuses attention on the algebraic form of the estimating functions and on the error criterion by which the approximation is judged. In [2] Fourier transform reconstructions were obtained from noise-free data samples by using best linear approximations in suitably chosen Hilbert spaces. These reconstructions included, as special cases, the estimates obtained by the bandlimited extrapolation procedures of Cadzow [3], Papoulis [4], Kolba and Parks [5], and others. The same approach was used in [6] to derive algorithms for tomographic reconstruction. In this report we consider the problem of Fourier transform estimation from data corrupted by additive noise and we analyze the reconstructions obtained in [2] as statistical estimators.

We begin with a summary of the techniques developed in [2] for the noise-free case. These methods are then extended to the case of noisy data, and bias and variance formulae are obtained. The special case of sinusoids in noise is considered, and the amplitude estimates obtained for this case are shown to be superior to the best linear unbiased estimates.

The general problem to be considered here is the reconstruction of

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{iwt}\, dt , \tag{1}$$

given the values at $t = t_1, \ldots, t_N$ of

$$x(t) = f(t) + n(t) . \tag{2}$$

Specific assumptions about $n(t)$, the unwanted or noise component, will be made as needed. By the noise-free case we will mean $x(t) = f(t)$.

APPROXIMATING A FOURIER TRANSFORM FROM NOISELESS DATA

The problem of estimating $F(w)$, given $f(t_1), \ldots, f(t_N)$, is often referred to as the quadrature problem; one must, in effect, perform the integration in Eq. (1) in some sense. Unless more than just the numerical data are known about $F(w)$ (or equivalently, about $f(t)$), any function $\hat{F}(w)$ such that

$$f(t_n) = \int_{-\infty}^{\infty} \hat{F}(w) e^{-iwt_n}\, dw/2\pi , \qquad n = 1, \ldots N , \tag{3}$$

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is a possible solution. The main idea of [2] is that one must use whatever prior knowledge one has to eliminate solutions as unsuitable and to select a single $\hat{F}(w)$ that stands the best chance, as far as one's prior information is concerned, of being the true $F(w)$. An effective way of doing this is by the use of weighted L^2-spaces and best linear approximation.

Let $F_o(w)$ be a prior estimate of $F(w)$ and let $P(w) > 0$ be a weighting function. We take as the estimate of $F(w)$ that function $\hat{F}(w)$ that minimizes the weighted error

$$\int_{-\infty}^{\infty} \left[|F_o(w) - \hat{F}(w)|^2 / P(w) \right] dw \ , \tag{4}$$

subject to the data constraints of Eq. (3). The solution is easily shown to be

$$\hat{F}(w) = F_o(w) + P(w) \sum_{m=1}^{N} z_m e^{iwt_m} \ , \tag{5}$$

where the coefficients $z_1, ..., z_N$ are determined by Eq. (3). In addition to being closest to F_o in the sense of Eq. (4), this estimate, $\hat{F}(w)$, is the best estimate of $F(w)$ having the algebraic form of Eq. (5) in the sense that the error

$$\int_{-\infty}^{\infty} \left[\left| F(w) - \left\{ F_o(w) + P(w) \sum_{m=1}^{N} a_m e^{iwt_m} \right\} \right|^2 / P(w) \right] dw \ , \tag{6}$$

is a minimum when $a_m = z_m$, $m = 1, ..., N$. Although the use of a nonzero $F_o(w)$ provides added flexibility when such a prior estimate is available, we will take $F_o(w) = 0$ in what follows. The estimator, Eq. (5) with $F_o(w) = 0$, was considered in [2], where it was referred to as the PDFT estimator.

If the data are evenly spaced d units apart and $P(w) = 1$ for $|w| \leq \pi/d$, $P(w) = 0$ otherwise, then Eq. (5) reduces to the DFT of the data. If we should happen to know that $F(w) = 0$ for $|w| > \sigma$, where $0 < \sigma < \pi/d$, and we take $P(w) = 1$, $|w| \leq \sigma$, $P(w) = 0$ otherwise, we obtain the estimate

$$\hat{F}_\sigma(w) = \begin{cases} \sum_{m=1}^{N} z_m e^{iwmd} \ , & |w| \leq \sigma \\ 0 \ , & |w| > \sigma \end{cases} \ , \tag{7}$$

where $z_1, z_2, ..., z_N$ are determined by Eq. (3). As was shown in [7] this estimate is the closed-form equivalent of those obtained through extrapolation algorithms in [3], [4], [5] and elsewhere. The estimator in Eq. (7) is quite unstable, based as it is on the assumption that the data correspond to a signal that is precisely σ-band-limited. It was in an attempt to reduce the sensitivity exhibited by $\hat{F}_\sigma(w)$ to out-of-band components that we were led to consider the estimators in Eq. (5). We found that by taking $P(w) = 1$ for $|w| \leq \sigma$, and $P(w) = \epsilon > 0$ for $\sigma < |w| \leq \pi/d$, where ϵ is a small positive number, the resulting PDFT estimator provided a marked improvement over $\hat{F}_\sigma(w)$. In effect, we make $P(w)$ share with the true $F(w)$ broad features, such as we know them a priori. Relative energy concentrations and known component distributions such as radar clutter, noise, and even delta functions, can be incorporated in $P(w)$. As we begin to distinguish components from one another and to

take them into account in $P(w)$, we pass into what may properly be called the noisy case and this is the subject of the next section.

STATISTICAL ESTIMATION OF FOURIER TRANSFORMS

Suppose now that $x(t) = f(t) + n(t)$ and that $x(t_1), \ldots, x(t_N)$ are given. We shall obtain a PDFT estimate of $X(w)$, the Fourier transform of $x(t)$, and then use it to derive an estimate of $F(w)$. We select weighting functions, $P_f(w) > 0$ and $P_n(w) > 0$ (which correspond to $f(t)$ and $n(t)$ respectively), take $P(w) = P_f(w) + P_n(w)$, and obtain

$$\hat{X}(w) = P(w) \sum_{m=1}^{N} z_m e^{iwt_m}, \tag{8}$$

as the estimate for $X(w)$. As a reasonable choice for the estimate of $F(w)$ we choose

$$\hat{F}(w) = \left[P_f(w)/P(w)\right] \hat{X}(w),$$

or

$$\hat{F}(w) = P_f(w) \sum_{m=1}^{N} z_m e^{iwt_m}, \tag{9}$$

where the $z_1, \ldots, z_N$ are chosen to make Eq. (8) consistent with the data. In the special case in which $P_f(w)P_n(w) = 0$ (disjoint spectra), it is clear that $\hat{X}(w)$ estimates both $F(w)$ and $N(w)$, the Fourier transform of $n(t)$. An example of this case was considered in [2] where Doppler radar samples were analyzed to detect targets known to be outside the clutter spectrum.

To compute the bias and variance of $\hat{F}(w)$ treated as a statistical estimator of $F(w)$, it is necessary to make further assumptions about $n(t)$. Before doing that, however, it is worth noting that up to now $n(t)$ has been any additive component that we desire to remove by filtering. If we take

$$\hat{f}(t) = \int_{-\infty}^{\infty} \hat{F}(w) e^{-iwt} dw/2\pi,$$

$$= \sum_{m=1}^{N} z_m p_f(t - t_m), \tag{10}$$

where $p_f(t)$ is the inverse Fourier transform of $P_f(w)$, then we have filtered $x(t)$. Indeed, by setting

$$\hat{f}(t_n) = \sum_{m=1}^{N} z_m p_f(t_n - t_m), \qquad n = 1, \ldots, N, \tag{11}$$

and then using these values as noiseless data in Eq. (5) (with $F_o(w) = 0, P(w) = P_f(w)$), we reproduce the estimate, $\hat{F}(w)$, of Eq. (9). The PDFT can in this way be used to introduce prior information into linear filtering.

From now on it will be assumed that $n(t)$ is one realization of a mean-zero weakly stationary random process, with power spectral density function, $R_n(w)$, and autocorrelation function, $r_n(t)$.

With $P(w) = P_f(w) + P_n(w)$ and $p(t)$ the inverse Fourier transform of $P(w)$, let G be the $N \times N$ matrix

$$G = [p(t_n - t_m)] \ , \tag{12}$$

and let C be its inverse

$$C = [C_{n,m}] = G^{-1} \ . \tag{13}$$

Then we can write

$$\hat{F}(w) = \sum_{m=1}^{N} H_m(w) x(t_m) \ , \tag{14}$$

with

$$H_m(w) = P_f(w) \sum_{n=1}^{M} C_{n,m} e^{iwt_m} \ , \qquad m = 1, ..., N \ . \tag{15}$$

It will be convenient to define, for any vector $v = (v_1, ..., v_N)$,

$$T(v,w) = \sum_{m=1}^{M} H_m(w) v_m \ . \tag{16}$$

The expected value of $\hat{F}(w)$ is then

$$E(\hat{F}(w)) = T(f,w) \ , \tag{17}$$

for

$$f = (f(t_1), ..., f(t_N)) \ ,$$

so that the bias is

$$F(w) - E(\hat{F}(w)) = F(w) - T(f,w) \ . \tag{18}$$

Note that $T(f,w)$ is the estimate of $F(w)$ that would be obtained if the data vector were the noiseless vector f. One source of bias is obviously traceable to deficiencies in the prior information even in the noiseless case (the quadrature problem). Another source of bias stems from the decision to design the estimator $\hat{F}(w)$ to operate in a noisy context and to use a prior estimate of the signal-to-noise ratio of roughly $P_f(w)/P_n(w)$. The bias in Eq. (18) depends solely on what would happen if the noise mysteriously vanished. We purposely introduce bias so that the variance can be reduced.

The variance of $\hat{F}(w)$ is given by

$$\mathrm{var}\,[\hat{F}(w)] = \sum_{m=1}^{N} \sum_{n=1}^{N} H_m(w) \overline{H_n(w)}\, p(t_m - t_n) \ , \tag{19}$$

or

$$\mathrm{var}\,[\hat{F}(w)] = \int_{-\infty}^{\infty} |T(e(\theta),w)|^2 R_n(\theta) d\theta / 2\pi \,, \tag{20}$$

where $e(\theta) = (\exp(-it_1\theta), ..., \exp(-it_N\theta))$. Note that $T(e(\theta),w)$ is the value of the estimator $\hat{F}(w)$ based on a data vector corresponding to samples of a pure sinusoid at frequency θ. To the extent that $P_f(w)$ effectively describes $F(w)$, it is to be expected that most of the $e(\theta)$ data will be attributed to noise, especially if $R_n(\theta)$ is large and this is reflected in $P_n(\theta)$. The spillover of noise at θ into signal estimated energy at w should be small.

Next we consider the special case of sinusoids in noise.

SINUSOIDS IN NOISE

Let the information-bearing component of the received signal consist of K sinusoids

$$f(t) = \sum_{k=1}^{K} b_k e^{-iw_k t} \,, \tag{21}$$

and, as before, let the given data be $x(t_n)$, $n = 1, ..., N$, where $x(t) = f(t) + n(t)$. The problem posed is to estimate the number of sinusoids present, K, their frequencies, $w_1, ..., w_K$, and their complex amplitudes, $b_1, ..., b_K$.

No single weighting function is suitable for determining both the locations, w_k, and the amplitudes, b_k, of the sinusoids. Before the number and location of the sinusoids are known, $P_f(w)$ should be chosen to be flat, corresponding to our knowledge that almost all of $F(w)$ is zero. The estimate, $\hat{F}(w)$, thus obtained then provides information about where spectral peaks are not to be found. Where peaks are located we can expect, using this flat $P_f(w)$, to see only moderate indications in $\hat{F}(w)$, because of the algebraic limitations of the estimating function (9). After a good estimate of the peak locations has been obtained, that information can be incorporated in the estimator by choosing a new weight, $P_f(w)$, which is larger in the neighborhood of likely sinusoidal peaks. In the limiting case, when the values of w_k are known exactly, a weighting function containing delta functions can be used to estimate the complex amplitudes, b_k.

To consider this limiting case in more detail, suppose the sinusoid frequencies, $w_1, ... w_k$, have been determined and $P_f(w)$ has been set equal to

$$P_f(w) = 2\pi \sum_{k=1}^{K} \delta(w - w_k) \,, \tag{22}$$

so that

$$p_f(t) = \sum_{k=1}^{K} e^{-iw_k t} \,. \tag{23}$$

The noise component is associated with a density $P_n(w)$, whose inverse transform, $p_n(t)$, we write as

$$p_n(t) = \propto g_n(t) ,$$

where

$$\propto = p_n(0) > 0 .$$

The matrix G in Eq. (12) becomes

$$\begin{aligned} G &= [p_f(t_j - t_m) + p_n(t_j - t_m)] \\ &= [p_f(t_j - t_m)] + \propto [g_n(t_j - t_m)] . \end{aligned}$$

Letting J be the $N \times K$ matrix $J = [\exp(-iw_k t_m)]$ we have, for $G_n = [g_n(t_j - t_m)]$,

$$G = JJ^* + \propto G_n . \tag{24}$$

Using Eq. (22) in Eq. (9) we can write $\hat{F}(w)$ as

$$\hat{F}(w) = 2\pi \sum_{k=1}^{K} \left(\sum_{m=1}^{N} z_m e^{iw_k t_m} \right) \delta(w - w_k) , \tag{25}$$

and, thus, from Eq. (21) the estimate for b_k is given by

$$\hat{b}_k = \sum_{m=1}^{N} z_m e^{iw_k t_m} , \tag{26}$$

where

$$z^T = (z_1, ..., z_N) = x^T G^{-1} , \tag{27}$$

and where

$$x^T = (x(t_1), ..., x(t_N)) .$$

Letting $\hat{b}^T = (\hat{b}_1, ..., \hat{b}_K)$, we obtain

$$\hat{b} = J^* z , \tag{28}$$

and

$$\begin{aligned} \hat{b} &= J^* G^{-1} x \\ &= J^* (JJ^* + \propto G_n)^{-1} x \\ &= J^* (G_n^{-1} JJ^* + \propto I)^{-1} G_n^{-1} x , \end{aligned} \tag{29}$$

where I is the identity matrix. A simple calculation shows that

$$J^* (G_n^{-1} JJ^* + \propto I)^{-1} = (\propto I + J^* G_n^{-1} J)^{-1} J^* , \tag{30}$$

and so we obtain,

$$\hat{b} = (\propto I + J^* G_n^{-1} J)^{-1} J^* G_n^{-1} x \,. \tag{31}$$

It follows that as $\propto$ converges to zero, our estimating vector $\hat{b}$ converges to

$$\hat{b}_o = (J^* G_n^{-1} J)^{-1} J^* G_n^{-1} x \,, \tag{32}$$

which is easily recognized as the best (minimum variance) linear unbiased estimate (BLUE) of the coefficients $b_1, ..., b_K$. By taking $\propto$ to zero, we gradually eliminate the noise component of the prior $P(w)$. As we saw earlier, this noise component is the source of bias, and so, in the limit when $\propto$ goes to zero we obtain an unbiased estimate. The estimate in Eq. (28) is biased to account for the assumed presence of a noise component at a level of $\propto$ relative to the signal energy level described by $P_f(w)$. If the prior assessment of SNR (signal-to-noise ratio) is accurate, it is to be expected that Eq. (28) will provide a better estimate than the BLUE.

The inversion of the matrix G in Eq. (24) is simplified by observing that JJ^* is a sum of outer products (column matrix times row matrix). An efficient scheme based on the identity

$$(A + xx^T)^{-1} = A^{-1} - \frac{(A^{-1}x)(x^T A^{-1})}{1 + x^T A x} \,,$$

for column vector x, has been used successfully in simulations we have run.

SUMMARY

We have described methods for the linear reconstruction of Fourier transforms from noiseless data and have extended them to the case of noisy samples. Bias and variance of these estimators in the presence of weakly stationary random noise were calculated and the special case of sinusoids in noise was considered. We found it useful to view this latter problem as one of frequency estimation followed by amplitude estimation. The resulting biased amplitude estimates were compared to their limiting unbiased values, which were shown to correspond to the BLUE.

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